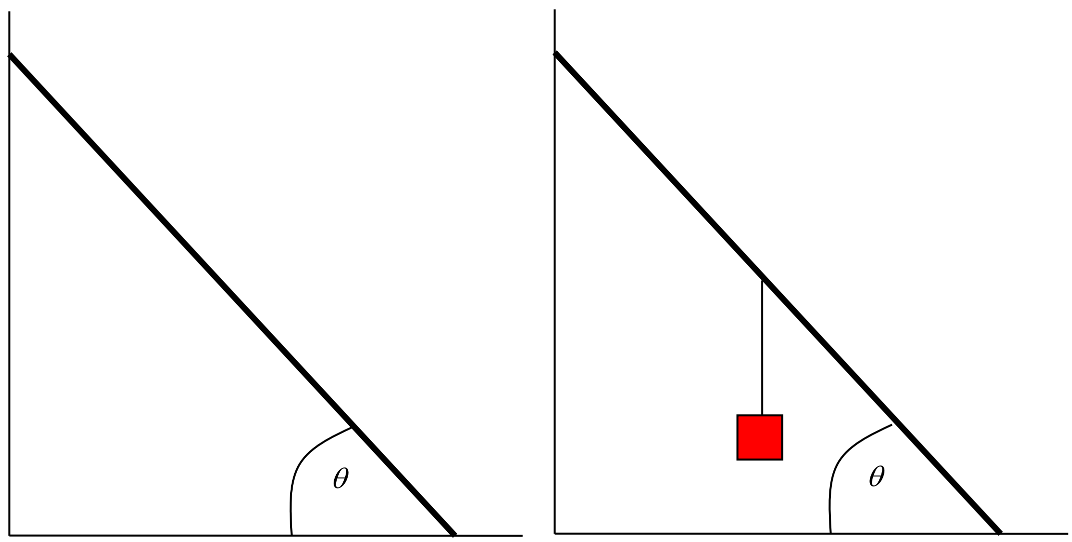
Teacher notes
Topic A

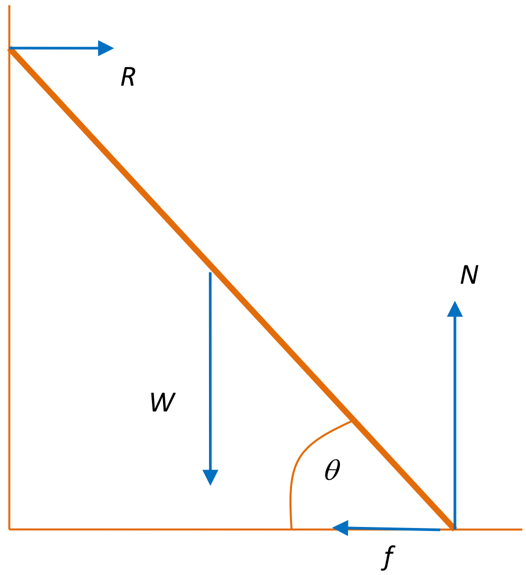
An equilibrium problem requiring intuition.

A rod of length L and weight W rests leaning on a frictionless vertical wall. The coefficient of static friction between the rod and the floor is μ. The minimum angle between the rod and the floor for which we have equilibrium is θ. A block of mass m is attached to a string and hangs from the middle of the rod. Will the rod still be in equilibrium making an angle θ with the floor?



Try to guess the answer without calculations first and start thinking of what happens when the point of support is somewhere else on the rod.

Without the hanging mass the forces on the rod are:



At equilibrium we have:

$N = W$

$f_{max} = R = \mu N$ (we use the maximum possible frictional force because we have been told that θ is the minimum angle for equilibrium thus requiring the maximum frictional force)

Taking torques about the point where the rod touches the floor:

$$RL\sin\theta = W\frac{L}{2}\cos\theta \Rightarrow R = \frac{W}{2\tan\theta}$$

Hence $f_{max} = \dfrac{W}{2\tan\theta} = \mu N = \mu W \Rightarrow \tan\theta = \dfrac{1}{2\mu}$.

Notice that the angle only depends on the coefficient of friction and not on the weight of the rod. Adding the hanging mass at the middle of the rod effectively increases the weight of the rod. So, since the angle is independent of the weight nothing changes and the rod remains in equilibrium at the same angle. Explicitly, we see this as follows: assuming equilibrium, we now have:

$N = W + mg$

$f = R$ (we do not write $f_{max} = \mu N$ because we do not know if we will need the maximum frictional force yet)

Taking torques about the point where the rod touches the floor:

$$RL\sin\theta = (W + mg)\frac{L}{2}\cos\theta \Rightarrow R = \frac{W + mg}{2\tan\theta}$$

Hence $f = \dfrac{W + mg}{2\tan\theta} = \dfrac{W + mg}{2 \times \dfrac{1}{2\mu}} = \mu(W + mg) = \mu N$.

In other words, we still have equilibrium since the required frictional force is in fact the maximum possible.

What if the hanging mass is attached somewhere else on the rod other than the middle?

You should be able to guess that if is attached to the right of the middle we will have equilibrium.

Indeed, assuming equilibrium we would have:

$$N = W + mg$$

$$f = R$$

Taking torques about the point where the rod touches the floor (*x* is the distance from the right-hand end of the rod of the point where we attach the mass):

$$RL\sin\theta = W\frac{L}{2}\cos\theta + mgx\cos\theta \Rightarrow R = \frac{W + mg\dfrac{2x}{L}}{2\tan\theta}$$

Hence $f = \dfrac{W + mg\dfrac{2x}{L}}{2\tan\theta} = \dfrac{W + mg\dfrac{2x}{L}}{2 \times \dfrac{1}{2\mu}} = \mu(W + mg\dfrac{2x}{L})$.

If $\dfrac{2x}{L} < 1$, i.e. $x < \dfrac{L}{2}$ (string attached to the right of the middle of the rod), then
$f < \mu(W + mg) = \mu N = f_{\max}$.

Since $f < f_{\max}$ we have equilibrium with a frictional force that is less than the maximum possible.

If on the other hand, $\dfrac{2x}{L} > 1$, i.e. $x > \dfrac{L}{2}$ (string attached to the left of the middle of the rod), then
$f > \mu(W + mg) = \mu N = f_{\max}$

This means that $f > f_{\max}$ and so equilibrium is not possible.

(The case $\dfrac{2x}{L} = 1$, i.e. $x = \dfrac{L}{2}$, was the original question and leads to equilibrium as we saw.)